Line integrals

Tuesday, March 30, 2021 11:12 AM

In single-var, derivatives and integrals are essentially opposites

$$\frac{df}{dx} = g$$

FTC: $\int_a^b g(x)\,dx = f(b) - f(a)$

So far we have the following in multivar

① Differentiation → partial differentiation

Given $f(x,y)$ (two inputs, one output)

have $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ or together:

gradient $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$ (two outputs? inputs)

Q/ Given ∇f (and maybe some initial cond $f(x_0, y_0) = z_0$ can we integrate?

We learned about a kind of integration in multivariable: takes a function $g(x,y)$ with 2 inputs and one output and a region R.

Then

$$\iint_R g(x,y)\,dx\,dy \text{ is a number}$$

(one output)

Want something like FTC:

ie, given $\vec{a} \in \mathbb{R}^2$ and $\vec{b} \in \mathbb{R}^2$

Want

$$f(\vec{b}) - f(\vec{a}) = \int_{\vec{a}}^{\vec{b}} \nabla f\, d?$$

if we write the RHS in terms of the double integration we learned, get 2 problems:

① ∇f has 2 outputs, not 1

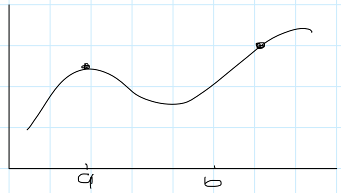
② How to choose R in terms of $\vec{a}$ and $\vec{b}$?

Need a new kind of integration s.t.

$$f(\vec{b}) - f(\vec{a}) = \int_{\vec{a}}^{\vec{b}} \nabla f\, d(\text{something})$$

this will be line integration [Co]138

Recall what happens in 1D



Q/ How to find $f(b)-f(a)$ in terms of $f'(x)$?

A/ Idea divide $[a,b]$ into little pieces $I_1, \dots, I_N$ eg $I_1 = \left[a, a+\frac{(b-a)}{N}\right]$

$$x_i = a + \frac{i(b-a)}{N} \quad \text{so} \quad I_i = [x_{i-1}, x_i]$$

$$\Delta x_i = \text{length}(I_i) = \frac{b-a}{N} = x_i - x_{i-1}$$

Q/ What is Δf over I_i?

A/ $f(x_i) - f(x_{i-1}) \leftarrow$ exact

$\Delta x_i \cdot f'(x_i) \leftarrow$ approx

↳ gets better as $N \to \infty$

$$f(b) - f(a) = f(x_N) - f(x_0) = \sum_i \Delta f$$

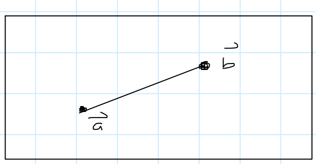
$$= \sum_i f(x_i) - f(x_{i-1})$$

$$= f(x_1) - f(x_0) + f(x_2) - f(x_1) + \dots + f(x_n) - f(x_{N-1})$$

$$\approx \sum_i \Delta x_i \cdot f'(x_i)$$

$\leftarrow$ Riemann sum whose $\lim_{N\to\infty}$ is $\int_a^b f'(x)\,dx$

Now, in 2D



$\mathbb{R}^2$

$$(x_i, y_i) = \vec{a} + \frac{i(\vec{b}-\vec{a})}{N}$$

Notice

$$(x_0, y_0) = \vec{a} \qquad (x_N, y_N) = \vec{b}$$

Now

$$f(\vec{b}) - f(\vec{a}) = f(x_N, y_N) - f(x_0, y_0)$$

$$= \sum_{i=1}^{N} f(x_i, y_i) - f(x_i, y_{i-1})$$

$$= \sum^{N} \Delta f$$

$$\sum_{i=1} f(x_i, y_i) - f(x_{i-1}, y_{i-1})$$

$$= \sum_{i=1}^{N} \underbrace{\Delta f}_{(x_{i-1},\, y_{i-1}) \to (x_i,\, y_i)}$$

Q/ How to estimate Δf using derivatives?

A/

<u>Notice</u> we have $\Delta x_i = x_i - x_{i-1}$

$\Delta y_i = y_i - y_{i-1}$

$$\Delta f \simeq \Delta x_i \frac{\partial f}{\partial x}(x_i, y_i) + \Delta y_i \frac{\partial f}{\partial y}(x_i, y_i)$$

So $$f(\vec{b}) - f(\vec{a}) = \sum_{i=1}^{N} \Delta f$$

$$\simeq \sum_{i=1}^{N} \Delta x_i \frac{\partial f}{\partial x} + \Delta y_i \frac{\partial f}{\partial y}$$

will define line integrals using lines like this

$$= \sum_{i=1}^{N} (\Delta x_i, \Delta y_i) \cdot \underbrace{\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial x}\right)}_{\nabla f}$$

$$= \sum_{i=1}^{N} \nabla f \cdot (\Delta x_i, \Delta y_i)$$

$$= \sum_{i=1}^{N} \nabla f \cdot \Delta \vec{r_i}$$

We call $$\lim_{N \to \infty} \sum_{i=1}^{\infty} \nabla f \cdot \Delta \vec{r_i} = \int_{\vec{a}}^{\vec{b}} \nabla f \cdot d\vec{r}$$

$$= \int_{\vec{a}}^{\vec{b}} \nabla f \cdot (dx, dy)$$

$$= \int_{\vec{a}}^{\vec{b}} \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \cdot (dx, dy)$$

$$= \int_{\vec{a}}^{\vec{b}} \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

In general, for a fcn w/ 2 outputs & inputs (like ∇f) we can define

$$(P(x,y), Q(x,y)) = (P, Q)$$

we can define $$\int_{\vec{a}}^{\vec{b}} (P, Q) \cdot d\vec{r} = \int_{\vec{a}}^{\vec{b}} P\,dx + Q\,dy$$

$$= \lim_{N \to \infty} \sum_{i=1}^{N} P(x_i, y_i) \cdot \Delta x_i + Q(x_i, y_i) \cdot \Delta y_i$$

When we defined $\int_{\vec{a}}^{\vec{b}} (P, Q) \cdot (dx, dy)$, we set

$$(x_i, y_i) = \vec{a} + \frac{i(\vec{b} - \vec{a})}{N}$$

such a point is on the line segment from $\vec{a}$ to $\vec{b}$.

More generally, we can integrate $(P, Q) \cdot (dx, dy)$ along any $\underbrace{\text{curve}}$ from $\vec{a}$ to $\vec{b}$.

↳ 1D subset of $\mathbb{R}^2$ that has a connected start and endpoint

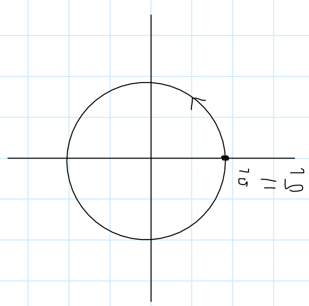
If $\vec{a} = \vec{b}$, we call a curve from $\vec{a}$ to $\vec{b}$ a loop

→ trivial loop stays at $\vec{a}$

Often, we parameterize a path, using t in some interval in $\mathbb{R}^1$

eg $\vec{a} = \vec{b} = (1, 0)$ in $\mathbb{R}^2$

consider the loop given by the unit circle (counterclockwise)



Q/ How to parameterize?

(You can parameterize same path in diff ways.

eg

$(x(t), y(t)) = (\cos(t), \sin(t))$
$t \in [0, 2\pi]$

$(x(t), y(t)) = (\cos(2\pi t), \sin(2\pi t))$
$t \in [0, 1]$

$(x, y) = (\cos(2\pi t^2), \sin(2\pi t^2))$
$t \in [0, 1]$

A/ Given a path C from $\vec{a}$ to $\vec{b}$, will define

$$\int_C (P, Q) \cdot (dx, dy)$$

To calculate it, we need to choose a parameterization of C, but the value of the integral is ind to the parameterization

Abstract definition of $\int_C P\,dx + Q\,dy$

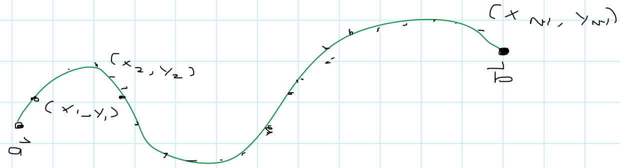
For each N, choose a partition of C into smaller paths.

Suppose C is from $\vec{a}$ to $\vec{b}$.

↳ set $(x_0, y_0) = \vec{a}$

$(x_N, y_N) = \vec{b}$

↳ Choose (x_1, y_1), (x_2, y_2), generally (x_i, y_i) on path C.



form Riemann sum:

$$\sum_{i=1}^{N} P(x_i, y_i)\cdot \Delta x_i + Q(x_i, y_i)\cdot \Delta y_i)$$

$\Delta x_i = x_i - x_{i-1}$

$\Delta y_i = y_i - y_{i-1}$

want as $N \to \infty$, the $\max(\Delta x_i)$ and ____ go to 0

equivalently: mesh = $\max \Delta x_i, \Delta y_i$

$$\int_C P\,dx + Q\,dy = \lim_{N\to\infty} \sum_{i=1}^{N} P(x_i, y_i)\Delta x_i + Q(x_i, y_i)\Delta y_i$$

$$= \lim_{\text{mesh}\to 0} \sum$$

$$\int_C P\,dx + Q\,dy = \int_C P\,dt\frac{dx}{dt} + Q\,dt\,\frac{dx}{dt}$$

$$= \int_{t=a}^{t=b} \left(P\,\frac{dx}{dt} + Q\,\frac{dy}{dt}\right) dt$$